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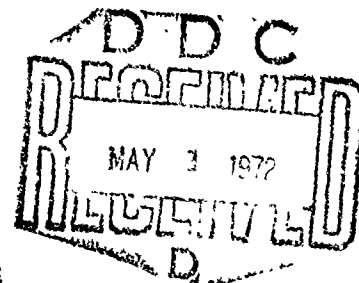
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THE TRANSMISSION OF FORCE BETWEEN

TWO HALF-PLANES

by

Ian N. Sneddon

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13. ABSTRACT

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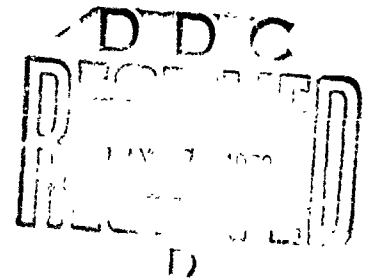
1. where one half-plane _rests_ on the other, maintaining contact along their common boundary;

2. where the two half-planes are bonded together;

3. where an imperfect bond leaves a Griffith crack at the interface;

4. where the lower half-plane is completely rigid.

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THE TRANSMISSION OF FORCE BETWEEN TWO HALF-PLANES

by

Ian N. Sneddon
(University of Glasgow)

April 14, 1972

PREFACE BY PROJECT DIRECTOR

This report by Dr. Sneddon presents some of his continuing research in the areas of elasticity, crack problems, and the applicability of transform methods.

The research project, PSR-99, at North Carolina State University is currently supported by Grant AFOSR-69-1779C.

Dr. Sneddon is Simson Professor of Mathematics at the University of Glasgow and is an Adjunct Professor at North Carolina State University.

The excellent work of Mrs. Dana Hall in the preparation of this report is gratefully acknowledged.

W. J. Harrington
Project Director

ABSTRACT

Consideration is given to the problem of determining the distribution of stress in a composite solid consisting of two half-planes (of different elastic moduli) joined together when there is a prescribed distribution of body forces acting in one of them. Included are the following cases:

1. where one half-plane <u>rests</u> on the other, maintaining contact along their common boundary;

2. where the two half-planes are <u>bonded</u> together;

3. where an imperfect bond leaves a Griffith crack at the interface;

4. where the lower half-plane is completely rigid.

THE TRANSMISSION OF FORCE BETWEEN TWO HALF-PLANES

by

Ian N. Sneddon

1. Introduction.

In this paper we consider the problem of determining the distribution of stress in a composite solid consisting of two half-planes (of different elastic moduli) joined together, when there is a prescribed distribution of body forces acting in one of them. Particular cases of the problem have been considered previously by Frasier and Rongved [1] and by Dunders [2].

It is assumed that in the upper half-plane $y > 0$ which is occupied by elastic material with rigidity modulus G_1 and Poisson's ratio η_1 there is a prescribed distribution of body forces, and that there is a displacement field $\{u_x^o(x, y),$ $u_y^o(x, y)\}$ which has the correct singularities to describe this distribution. We consider the case of plane strain in which it is natural to take G_1 and $\kappa_1 = 3 - 4\eta_1$ as the elastic constants. The advantage in this choice of constants is that the results for plane stress take exactly the same form except that in this case $\kappa_1 = (3 - \eta_1)/(1 + \eta_1)$. The lower half-plane $y < 0$ is assumed to be occupied by an elastic material with constants G_2 and κ_2.

In §3 we consider the situation in which one half-plane rests on the other and derive formulae for the calculation of the displacement and stress fields in terms of the prescribed displacement vector $\underset{\sim}{u}^o$. It is assumed that the two half-planes remain in contact along the entire length of their common boundary. These formulae take much simpler forms if the displacement vector $\underset{\sim}{u}^o$, which is arbitrary apart from the fact that it must have the right kind of singularities to account from the prescribed distribution of body forces, is chosen in such a way that $u_y^o(x, 0) \equiv 0$ and $\sigma_{xy}^o(x, 0) \equiv 0$ (This can often be achieved by the use

of an "image" method of the kind represented pictorially in Fig. 1). In this case, for instance, we obtain the formula

$$\sigma_{yy}(x,\ 0-) = \sigma_{yy}(x,\ 0+) = D\sigma_{yy}^{o}(x,\ 0) \tag{1.1}$$

where D is the constant defined by the equation

$$D = \frac{(\kappa_1 + 1)\Gamma}{(\kappa_1 + 1)\Gamma + (\kappa_2 + 1)} \qquad (\Gamma = G_2/G_1), \tag{1.2}$$

To illustrate the use of the formulae we consider the problem of calculating the stress field due to a point force (X, -Y), (X > 0, Y > 0) acting at the point (0, c), c > 0, in the upper half-plane.

In §4 we consider the situation in which the two half-planes are __bonded__ together. The formulae are now much more complicated. For instance, even in the symmetrical case in which $u_y^o(x,\ 0) \equiv 0$ and $\sigma_{xy}^o(x,\ 0) \equiv 0$ the formula corresponding to (1.1) is of the form

$$\sigma_{yy}(x,\ 0-) = D_1\ \sigma_{yy}^o(x,\ 0) - D_2 G_2\ \frac{\partial u_x(x,\ 0)}{\partial x}$$

where D_1 and D_2 are numerical constants (cf. equation (4.5) below). Again, the method is illustrated by deriving the formula appropriate to the case in which a point force acts in the upper half-plane.

In §5 we consider the situation in which the bonding between the two half-planes is not perfect but leaves a Griffith crack at the interface of the two half-planes. It is shown that the solution of the problem in which the crack is opened out by the application of prescribed internal pressure can be reduced to that of a set of four simultaneous dual integral equations.

Finally, in §6 we discuss the special case in which the lower half-plane is completely rigid.

2. The Basic Solutions of the Equilibrium Equations in the Case of Plane Strain.

By the use of the Fourier transform (see, e.g., [3], p. 404) we can show that the equations of plane strain have solution

$$u_x(x, y) = iF*[\xi^{-1}\{(\kappa + 1)\frac{\partial^2 X}{\partial y^2} + (3 - \kappa)\xi^2 X\} ; \xi \to x] \tag{2.1}$$

$$u_y(x, y) = F*[\xi^{-2}\{(\kappa + 1)\frac{\partial^3 X}{\partial y^3} - (5 + \kappa)\xi^2 \frac{\partial X}{\partial y} ; \xi \to x] \tag{2.2}$$

where the function $X(\xi, y)$ satisfies the equation

$$\left(\frac{\partial^2}{\partial y^2} - \xi^2\right)^2 X(\xi, y) = 0. \tag{2.3}$$

The constant κ is defined in terms of the Poisson ratio η through the equation

$$\kappa = 3 - 4\eta \tag{2.4}$$

and $F*$ denotes the operator defined by the equation

$$F*[f(\xi, y); \xi \to x] = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(\xi, y)e^{-i\xi x} d\xi \tag{2.5}$$

i.e. it is the inverse of the operator F defined by

$$F[\psi(x, y); x \to \xi] = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \psi(x, y)e^{i\xi x} dx. \tag{2.6}$$

If we take for X the function

$$X(\xi, y) = \frac{1}{4}\kappa^{-1} |\xi|^{-2}\{\frac{1}{2}(\kappa - 1)A + \frac{1}{2}(\kappa + 1)B - (A - B)|\xi|y\}e^{-|\xi|y} \tag{2.7}$$

we obtain the displacement field

$$u_x(x, y) = F*[i\xi^{-1}\{ \} - \kappa^{-1}(A - B)|\xi|y\}e^{-|\xi|y}; \xi \to x] \tag{2.8}$$

$$u_y(x, y) = F*[|\xi|^{-1}\{B - \kappa^{-1}(A - B)|\xi|y\}e^{-|\xi|y}; \xi \to x] \tag{2.9}$$

which is such that

$$u_x(x, 0) = F*[i\xi^{-1}A(\xi); x] \tag{2.10}$$

$$u_y(x, 0) = F*[|\xi|^{-1}B(\xi); x] \tag{2.11}$$

$$\sigma_{xy}(x, 0) = -\kappa^{-1}G \ F*[i \ \text{sgn} \ \xi\{(\kappa + 1)A(\xi) + (\kappa - 1)B(\xi)\}; x] \tag{2.12}$$

$$\sigma_{yy}(x, 0) = -\kappa^{-1}G \ F*[(\kappa - 1)A(\xi) + (\kappa + 1)B(\xi); x]. \tag{2.13}$$

In the two latter equations G denotes the rigidity modulus.

On the other hand, if we take for X the function

$$X(\xi, y) = \frac{1}{4}\kappa^{-1}|\xi|^{-2}\{\frac{1}{2}(\kappa - 1)A - \frac{1}{2}(\kappa + 1)B + (A + B)|\xi|y\}e^{|\xi|y} \tag{2.14}$$

we obtain the displacement field

$$u_x(x, y) = F*[i\xi^{-1}\{A + \kappa^{-1}(A + B)|\xi|y\}e^{|\xi|y}; \xi \to x] \tag{2.15}$$

$$u_y(x, y) = F*[|\xi|^{-1}\{B - \kappa^{-1}(A + B)|\xi|y\}e^{|\xi|y}; \xi \to x] \tag{2.16}$$

which is such that

$$u_x(x, 0) = F*[i\xi^{-1}A(\xi); x] \tag{2.17}$$

$$u_y(x, 0) = F*[|\xi|^{-1}B(\xi); x] \tag{2.18}$$

$$\sigma_{xy}(x, 0) = \kappa^{-1}G \ F*[i \ \text{sgn} \ \xi\{(\kappa + 1)A(\xi) - (\kappa - 1)B(\xi)\}; x] \tag{2.19}$$

$$\sigma_{yy}(x, 0) = -\kappa^{-1}G \ F*[(\kappa - 1)A(\xi) - (\kappa + 1)B(\xi); x]. \tag{2.20}$$

3. One Half-Plane Resting on Another.

We begin by considering the case where one half-space _rests_ upon the other the loading being assumed to be such that they always remain completely in contact. The half-space $y > 0$ is assumed to be occupied by material with rigidity modulus G_1 and Poisson's ratio η_1, while the half-space $y < 0$ is assumed to be occupied by material with rigidity modulus G_2 and Poisson's ratio η_2. In conformity with equation (2.4) we write $\kappa_1 = 3 - 4\eta_1$, $\kappa_2 = 3 - 4\eta_2$. The boundary conditions in this case are

$$u_y(x, 0+) = u_y(x, 0-) \tag{3.1}$$

$$\sigma_{xy}(x, 0+) = 0, \ \sigma_{xy}(x, 0-) = 0 \tag{3.2}$$

$$\sigma_{yy}(x, 0+) = \sigma_{yy}(x, 0-). \tag{3.3}$$

We suppose that in the half-space $y > 0$ there is a distribution of body forces and that there is a displacement field $\{u_x^o(x, y),\ u_y^o(x, y)\}$ in $y > 0$ which has the correct singularities to describe this distribution. From equations (2.8), (2.9), (2.15), (2.16) we see that we can describe the displacement field in the composite solid by the equations

$$u_x(x, y) = \begin{cases} u_x^o(x, y) + iF^*[\xi^{-1}\{A_1 - \kappa_1^{-1}(A_1 - B_1)|\xi|y\}e^{-|\xi|y}; \ \xi \to x], & y > 0; \\[2mm] iF^*[\xi^{-1}\{A_2 + \kappa_2^{-1}(A_2 + B_2)|\xi|y\}e^{|\xi|y}; \ \xi \to x], & y < 0; \end{cases}$$

$$u_y(x, y) = \begin{cases} u_y^o(x, y) + F^*[|\xi|^{-1}\{B_1 - \kappa_1^{-1}(A_1 - B_1)|\xi|y\}e^{-|\xi|y}; \ \xi \to x], & y > 0; \\[2mm] F^*[|\xi|^{-1}\{B_2 - \kappa_2^{-1}(A_2 + B_2)|\xi|y\}e^{|\xi|y}; \ \xi \to x], & y < 0. \end{cases}$$

The boundary conditions (3.1) through (3.3) then lead to the relations

$$B_2(\xi) = B_1(\xi) + |\xi|\hat{v}(\xi)$$

$$(\kappa_1 + 1)A_1(\xi) + (\kappa_1 - 1)B_1(\xi) = -i\kappa_1 G_1^{-1}\hat{\tau}(\xi)$$

$$(\kappa_2 + 1)A_2(\xi) - (\kappa_2 - 1)B_2(\xi) = 0$$

$$\kappa_2^{-1}G_2\{(\kappa_2 - 1)A_2(\xi) - (\kappa_2 + 1)B_2(\xi)\} = \kappa_1^{-1}G_1\{(\kappa_1 - 1)A_1(\xi) + (\kappa_1 + 1)B_1(\xi)\} - \hat{\sigma}(\xi)$$

connecting the unknown functions $A_1(\xi)$, $A_2(\xi)$, $B_1(\xi)$, $B_2(\xi)$ with the known functions $\hat{u}(\xi)$, $\hat{\tau}(\xi)$, $\hat{\sigma}(\xi)$ defined by the equations

$$\hat{v}(\xi) = F[u_y^o(x, 0); \xi]$$

$$\hat{\tau}(\xi) = F[\sigma_{xy}^o(x, 0); \xi] \tag{3.4}$$

$$\hat{\sigma}(\xi) = F[\sigma_{yy}^o(x, 0); \xi].$$

We may write the solution of these equations in the form

$$A_1(\xi) = -\frac{\kappa_1 - 1}{\kappa_1 + 1}\{(\kappa_2 + 1)f(\xi) - D|\xi|\hat{v}(\xi)\} - i\kappa_1 G_1^{-1}\hat{\tau}(\xi) \tag{3.5}$$

$$B_1(\xi) = (\kappa_2 + 1)f(\xi) - D|\xi|\hat{v}(\xi) \tag{3.6}$$

$$A_2(\xi) = (\kappa_2 - 1)f(\xi) - \frac{\kappa_2 - 1}{\kappa_2 + 1}(D - 1)|\xi|\hat{v}(\xi) \tag{3.7}$$

$$B_2(\xi) = (\kappa_2 + 1)f(\xi) - (D - 1)|\xi|\hat{v}(\xi) \tag{3.8}$$

where the function f is defined by the equation

$$f(\xi) = \frac{1}{4}DG_2^{-1}\{\hat{\sigma}(\xi) + (\kappa_1 - 1)\hat{\tau}(\xi)\} \tag{3.9}$$

and the constants Γ and D by the equations

$$\Gamma = \frac{G_2}{G_1}, \quad D = \frac{(\kappa_1 + 1)\Gamma}{(\kappa_1 + 1)\Gamma + (\kappa_2 + 1)}. \tag{3.10}$$

It follows immediately from equations (2.18) and (2.20) that

$$u_y(x, 0-) = \frac{1}{4}(\kappa_2 + 1)Dw(x) - (D - 1)u_y^o(x, 0+) \tag{3.11}$$

$$\sigma_{yy}(x, 0-) = D[\sigma_{yy}^o(x, 0+) + (\kappa_1 - 1)\sigma_{xy}^o(x, 0+) - s(x)] \tag{3.12}$$

where the functions $w(x)$ and $s(x)$ are defined by the equations

$$w(x) = G_2^{-1}F*[|\xi|^{-1}\{\hat{\sigma}(\xi) + (\kappa_1 - 1)\hat{\tau}(\xi)\}; x] \tag{3.13}$$

$$s(x) = \frac{4G_2}{\kappa_2 + 1} F*[|\xi|\hat{v}(\xi); x], \tag{3.14}$$

When $u_y^o(x, 0)$, $\sigma_{xy}^o(x, 0)$, $\sigma_{yy}^o(x, 0)$ are known we can calculate $w(x)$ and $s(x)$ from these equations. We may derive a _formal_ expression for $s(x)$ as follows:

Since

$$F[\sqrt{(2/\pi)}x^{-1}; \xi] = i \, \text{sgn} \, \xi$$

it follows that

$$F*[i \, \text{sgn} \, \xi \, \hat{v}(\xi); x] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)dt}{x - t},$$

Also

$$F*[|\xi|\hat{v}(\xi); x] = \frac{d}{dx} F*[i \, \text{sgn} \, \xi \, \hat{v}(\xi); x]$$

so that we obtain the formula

$$s(x) = \frac{1}{\pi} \frac{d}{dx} \int_{-\infty}^{\infty} \frac{u_y^o(t, 0)dt}{x - t}, \tag{3.15}$$

In certain cases it is possible to choose the vector (u_x^o, u_y^o) in such a way that for any values of x

$$u_y^o(x,\ 0) \equiv 0,\ \sigma_{xy}^o(x,\ 0) \equiv 0, \tag{3.16}$$

when

$$\hat{v}(\xi) \equiv 0,\ \hat{\tau}(\xi) \equiv 0$$

for all values of ξ. In that case the equations (3.5) through (3.9) are replaced by

$$A_1(\xi) = -\frac{(\kappa_1 - 1)(\kappa_2 + 1)}{4(\kappa_1 + 1)} DG_2^{-1}\hat{\sigma}(\xi) \tag{3.17}$$

$$B_1(\xi) = \frac{1}{4}(\kappa_2 + 1)DG_2^{-1}\hat{\sigma}(\xi) \tag{3.18}$$

$$A_2(\xi) = \frac{1}{4}(\kappa_2 - 1)DG_2^{-1}\hat{\sigma}(\xi) \tag{3.19}$$

$$B_2(\xi) = \frac{1}{4}(\kappa_2 + 1)DG_2^{-1}\hat{\sigma}(\xi) \tag{3.20}$$

and the displacement field is given by the equations

$$u_x(x,\ y) = \begin{cases} u_x^o(x,\ y) - \dfrac{(\kappa_2 + 1)D}{4(\kappa_1 + 1)}\ G_2^{-1}F*[i\xi^{-1}\{\kappa_1 - 1 - 2|\xi|y\}\hat{\sigma}(\xi)e^{-|\xi|y};\xi\to x], & (y > 0) \\[2em] \frac{1}{4}DG_2^{-1}F*[i\xi^{-1}\{\kappa_2 - 1 + 2|\xi|y\}\hat{\sigma}(\xi)e^{|\xi|y};\ \xi \to x] & (y < 0) \end{cases}$$

$$u_y(x,\ y) = \begin{cases} u_y^o(x,\ y) + \dfrac{(\kappa_2 + 1)D}{4(\kappa_1 + 1)}\ G_2^{-1}F*[|\xi|^{-1}\{\kappa_1 + 1 + 2|\xi|y\}\hat{\sigma}(\xi)e^{-|\xi|y};\xi\to x], & (y > 0) \\[2em] \frac{1}{4}DG_2^{-1}F*[|\xi|^{-1}\{\kappa_2 + 1 - 2|\xi|y\}\hat{\sigma}(\xi)e^{|\xi|y};\ \xi \to x] & (y < 0), \end{cases}$$

The corresponding expressions for components of the stress tensor are

$$\sigma_{xx}(x, y) = \begin{cases} \sigma_{xx}^{Q}(x, y) - (1 - D)F^*[(1 - |\xi|y)e^{-|\xi|y}\hat{\sigma}(\xi); \ \xi \to x], & (y > 0) \quad (3.21) \\ \\ DF^*[(1 + |\xi|y)e^{|\xi|y}\hat{\sigma}(\xi); \ \xi \to x], & (y < 0) \end{cases}$$

$$\sigma_{xy}(x, y) = \begin{cases} \sigma_{xy}^{0}(x, y) - (1 - D)y \ F^*[i\xi e^{-|\xi|y}\hat{\sigma}(\xi); \ \xi \to x], & (y > 0) \quad (3.22) \\ \\ yDF^*[i\xi e^{|\xi|y}\hat{\sigma}(\xi); \ \xi \to x] & (y < 0) \end{cases}$$

$$\sigma_{yy}(x, y) = \begin{cases} \sigma_{yy}^{0}(x, y) - (1 - D)F^*[(1 + |\xi|y)e^{-|\xi|y}\hat{\sigma}(\xi); \ \xi \to x] & (y > 0) \quad (3.23) \\ \\ DF^*[(1 - |\xi|y)e^{|\xi|y}\hat{\sigma}(\xi); \ \xi \to x] & (y < 0). \end{cases}$$

It should be noted that

$$\sigma_{yy}(x, 0-) = \sigma_{yy}(x, 0+) = D\sigma_{yy}^{0}(x, 0+) .$$

As a particular case we consider the stress field due to a point force $(X, -Y)$, $X > 0$, $Y > 0$ acting at the point $(0, c)$, $c > 0$ in the upper-half plane (cf. Fig. 1). It is obvious
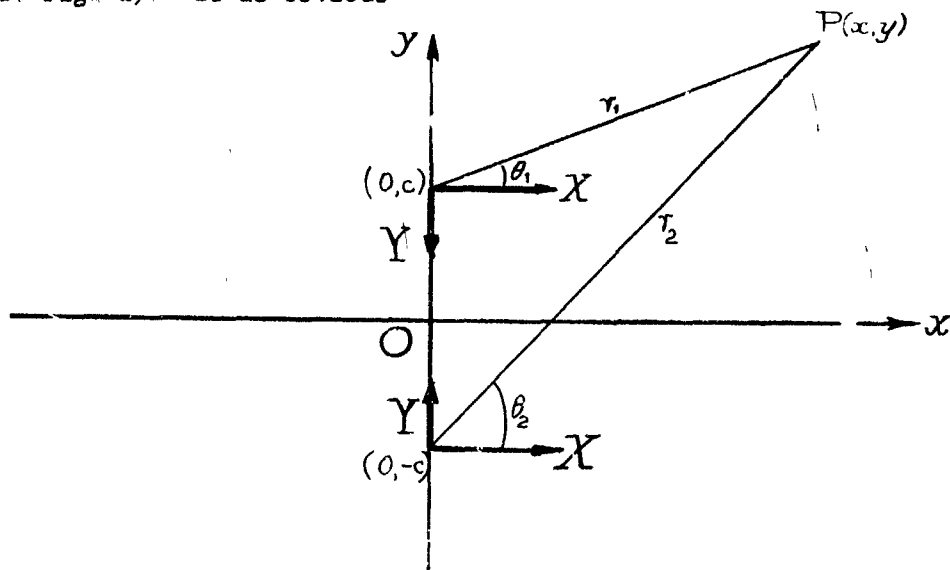

Fig 1

from symmetry considerations that if we consider the displacement field due to this point force and to the point force (X, Y) at (0, -c) we get a field which satisfies the conditions (3.16).

Using the well-known expression for the displacement due to a point force (see, e.g., Love, 1944, p. 209) we can easily show that the components of this displacement field are given by the equations

$$u^o_x(x, y) = - \frac{X}{2\pi(\kappa_1 + 1)G_1} \left[\kappa_1 \log(r_1 r_2) + \frac{(y - c)^2}{r_1^2} + \frac{(y + c)^2}{r_2^2} \right]$$

$$- \frac{xY}{2\pi(\kappa_1 + 1)G_1} \left[\frac{y - c}{r_1^2} - \frac{y + c}{r_2^2} \right] \tag{3.24}$$

$$u^o_y(x, y) = \frac{xY}{2\pi(\kappa_1 + 1)G_1} \left[\frac{y - c}{r_1^2} + \frac{y + c}{r_2^2} \right]$$

$$+ \frac{Y}{2\pi(\kappa_1 + 1)G_1} \left[\kappa_1 \log(r_1/r_2) + x^2 \left(\frac{1}{r_1^2} - \frac{1}{r_2^2} \right) \right] \tag{3.25}$$

where r_1 and r_2 are defined by the equations

$$r_1^2 = x^2 + (y - c)^2, \quad r_2^2 = x^2 + (y + c)^2.$$

From these expressions we in turn deduce that the components of the stress tensor are given by the equations

$$\sigma^o_{xx}(x, y) = -\frac{xX}{2\pi(\kappa_1+1)}\left[(\kappa_1+3)\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}\right) - 4\left\{\frac{(y-c)^2}{r_1^4}+\frac{(y+c)^2}{r_2^4}\right\}\right]$$

$$-\frac{Y}{2\pi(\kappa_1+1)}\left[(\kappa_1-1)\left(\frac{y-c}{r_1^2}-\frac{y+c}{r_2^2}\right) - 4x^2\left\{\frac{y-c}{r_1^4}-\frac{y+c}{r_2^4}\right\}\right] \quad (3.26)$$

$$\sigma^o_{xy}(x, y) = -\frac{X}{2\pi(\kappa_1+1)}\left[(\kappa_1-1)\left\{\frac{y-c}{r_1^2}+\frac{y+c}{r_2^2}\right\} + 4x^2\left\{\frac{y-c}{r_1^4}+\frac{y+c}{r_2^4}\right\}\right]$$

$$+\frac{xY}{2\pi(\kappa_1+1)}\left[(\kappa_1-1)\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right) + 4\left\{\frac{(y-c)^2}{r_1^4}-\frac{(y+c)^2}{r_2^4}\right\}\right] \quad (3.27)$$

$$\sigma^o_{yy}(x, y) = \frac{xX}{2\pi(\kappa_1+1)}\left[(\kappa_1-1)\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}\right) - 4\left\{\frac{(y-c)^2}{r_1^4}+\frac{(y+c)^2}{r_2^4}\right\}\right]$$

$$+\frac{Y}{2\pi(\kappa_1+1)}\left[(\kappa_1+3)\left(\frac{y-c}{r_1^2}-\frac{y+c}{r_2^2}\right) - 4x^2\left\{\frac{y-c}{r_1^4}-\frac{y+c}{r_2^4}\right\}\right], \quad (3.28)$$

In particular we confirm that the conditions (3.16) are satisfied and that

$$\sigma^o_{yy}(x, 0) = \frac{X}{\pi(\kappa_1+1)}\left[(\kappa_1-1)\frac{x}{x^2+c^2} - \frac{4c^2x}{(x^2+c^2)^2}\right]$$

$$-\frac{Y}{\pi(\kappa_1+1)}\left[(\kappa_1+3)\frac{c}{x^2+c^2} - \frac{4cx^2}{(x^2+c^2)^2}\right] \quad (3.29)$$

whose Fourier transform is readily shown to be

$$\hat{\sigma}(\xi) = \frac{X}{\sqrt{(2\pi)}(\kappa_1+1)}\left[(\kappa_1-1)i\,\text{sgn}\,\xi - 2i c\xi\right]e^{-c|\xi|}$$

$$-\frac{Y}{\sqrt{(2\pi)}(\kappa_1+1)}\left[(\kappa_1+1) + 2c|\xi|\right]e^{-c|\xi|}. \quad (3.30)$$

It should be noticed that the value of the ratio X/Y cannot be chosen
arbitrarily in this problem since the two half-planes will remain in contact
only if $\sigma_{yy}(x, 0) < 0$ for all real values of x, i.e. only if

$$(\kappa_1 + 3) + (\kappa_1 - 1)t^2 - (X/Y)t\{(\kappa_1 - 1)t^2 - (5 - \kappa_1)\} > 0 \tag{3.31}$$

for all real values of t.

The condition (3.31) is obviously satisfied in the case in which $X = 0$.
The equations (3.21) through (3.23) then yield the equations

$$\sigma_{xx}(x, y) = \begin{cases} \sigma_{xx}^0(x, y) - (1 - D)\sigma_{xx}^1(x, y) , & (y > 0) \\[2em] D\sigma_{xx}^2(x, y) , & (y < 0), \end{cases} \tag{3.32}$$

$$\sigma_{xy}(x, y) = \begin{cases} \sigma_{xx}^0(x, y) - (1 - D)\sigma_{xy}^1(x, y) , & (y > 0) \\[2em] D\sigma_{xy}^2(x, y) , & (y < 0), \end{cases} \tag{3.33}$$

$$\sigma_{yy}(x, y) = \begin{cases} \sigma_{yy}^0(x, y) - (1 - D)\sigma_{yy}^1(x, y) & (y > 0) \\[2em] D\sigma_{yy}^2(x, y) & (y < 0), \end{cases} \tag{3.34}$$

where with the notation

$$x + i(y - c) = r_1 e^{i\theta_1}, \quad x + i(y + c) = r_2 e^{i\theta_2} \tag{3.35}$$

we have

$$\sigma_{xx}^1 = -\frac{Y}{\pi(\kappa_1 + 1)r_2} [2(\kappa_1 + 1)\sin\theta_2\cos\theta_2 - \frac{c}{r_2}\{(\kappa_1 - 3)\cos2\theta_2 + 2\cos4\theta_2\} - \frac{4c^2}{r_2^2}\sin3\theta_2]$$

$$\sigma_{xy}^1 = -\frac{Yy}{\pi(\kappa_1 + 1)r_2^2} [(\kappa_1 + 1)\sin2\theta_2 - \frac{4c}{r_2}\cos3\theta_2]$$

$$\sigma_{yy}^1 = -\frac{Y}{\pi(\kappa_1 + 1)r_2} [2(\kappa_1 + 1)\sin^3\theta_2 + \frac{c}{r_2}\{(\kappa_1 + 1)\cos2\theta_2 + 2\cos4\theta_2\} + \frac{4c^2}{r_2^2}\sin3\theta_2]$$

$$\sigma_{xx}^2 = \frac{Y}{\pi(\kappa_1 + 1)r_1} [2(\kappa_1 + 1)\sin\theta_1 \cos^2\theta_1 + \frac{c}{r_1}\{(\kappa_1 + 1)\cos2\theta_1 + 2\cos4\theta_1\} - \frac{4c^2}{r_1^2}\sin3\theta_1]$$

$$\sigma_{xy}^2 = \frac{Yy}{\pi(\kappa_1 + 1)r_1^2} [(\kappa_1 + 1)\sin2\theta_1 + \frac{4c}{r_2}\cos3\theta_1]$$

$$\sigma_{yy}^2 = \frac{Y}{\pi(\kappa_1 + 1)r_1} [2(\kappa_1 + 1)\sin^3\theta_1 - \frac{c}{r_1}\{(\kappa_1 - 3)\cos2\theta_1 + 2\cos4\theta_1\} + \frac{4c^2}{r_1^2}\sin3\theta_1].$$

4. Two Half-Planes Bonded Together.

If the two half-planes are bonded together the conditions (3.1) through (3.3) are replaced by the condition that both the displacement vector and the stress tensor must be continuous across $y = 0$.

Using the same form for the displacement vector as before we find that the continuity of $u_x(x, y)$ and $u_y(x, y)$ on $y = 0$ gives the pair of equations

$$A_2(\xi) = A_1(\xi) + i\xi\hat{u}(\xi) \tag{4.1}$$

$$B_2(\zeta) = B_1(\xi) + |\xi|\hat{v}(\xi) \tag{4.2}$$

where $v(\xi)$ is defined in (3.4) and

$$\hat{u}(\xi) = F[u_x^o(x, 0); \xi]$$

Similarly the continuity of the stress components $\sigma_{xy}(x, 0)$, $\sigma_{yy}(x, 0)$ gives the pair of equations

From these expressions and the formula

$$F[\sigma_{yy}(x, 0-); \xi] = -\kappa_2^{-1}G_2[(\kappa_2 - 1)A_2(\xi) - (\kappa_2 + 1)B_2(\xi)]$$

we deduce that

$$F[\sigma_{yy}(x, 0-); \xi] = \frac{1}{2}\left[\frac{\kappa_1}{1 + \kappa_1\Gamma} + \frac{1}{\kappa_2 + \Gamma}\right]\Gamma\hat{\sigma}(\xi) + \frac{1}{2}\left[\frac{\kappa_1}{1 + \kappa_1\Gamma} - \frac{1}{\kappa_2 + \Gamma}\right]\Gamma\,\mathrm{isgn}\xi\hat{\tau}(\xi)$$

$$+ \frac{1}{4}G_2\left[\frac{1}{1 + \kappa_1\Gamma} - \frac{1}{\kappa_2 + \Gamma}\right]i\xi\hat{u}(\xi) + \frac{1}{4}G_2\left[\frac{1}{1 + \kappa_1\Gamma} + \frac{1}{\kappa_2 + \Gamma}\right]|\xi|\hat{v}(\xi).$$

Similarly from the equation

$$F[\sigma_{xy}(x, 0-); \xi] = \kappa_2^{-1}G_2\,\mathrm{isgn}\xi\{(\kappa_2 + 1)A_2(\xi) - (\kappa_2 - 1)B_2(\xi)\}$$

we deduce that

$$F[\sigma_{xy}(x, 0-); \xi] = -\frac{1}{2}\left[\frac{\kappa_1}{1 + \kappa_1\Gamma} - \frac{1}{\kappa_2 + \Gamma}\right]\Gamma\,\mathrm{isgn}\,\xi\hat{\sigma}(\xi) + \frac{1}{2}\left[\frac{\kappa_1}{1 + \kappa_1\Gamma} + \frac{1}{\kappa_2 + \Gamma}\right]\Gamma\tau(\xi)$$

$$-\frac{1}{4}G_2\left[\frac{1}{1 + \kappa_1\Gamma} + \frac{1}{\kappa_2 + \Gamma}\right]i\xi\hat{u}(\xi) - \frac{1}{4}G_2\left[\frac{1}{1 + \kappa_1\Gamma} - \frac{1}{\kappa_2 + \Gamma}\right]|\xi|\hat{v}(\xi).$$

The expression for $F[u_y(x, 0-); \xi]$ can be written down directly from equation (4.4) since by (2.18)

$$F[u_y(x, 0-); \xi] = |\xi|^{-1}B_2(\xi).$$

If the conditions (3.16) are satisfied we find that

$$\sigma_{yy}(x, 0-) = \frac{1}{2}\Gamma\left[\frac{\kappa_1}{1 + \kappa_1\Gamma} + \frac{1}{\kappa_2 + \Gamma}\right]\sigma_{yy}^0(x, 0+)$$

$$-\frac{1}{4}\left[\frac{1}{1 + \kappa_1\Gamma} - \frac{1}{\kappa_2 + \Gamma}\right]G_2\frac{\partial u_x^0(x, 0)}{\partial x} \tag{4.5}$$

$$\sigma_{xy}(x, \ 0-) = \frac{1}{4}\left[\frac{1}{1 + \kappa_1\Gamma} + \frac{1}{\kappa_2 + \Gamma}\right]G_2 \ \frac{\partial u_x^o(x, \ 0)}{\partial x}$$

$$-\frac{1}{2}\Gamma\left[\frac{\kappa_1}{1 + \kappa_1\Gamma} - \frac{1}{\kappa_2 + \Gamma}\right]F^*[\text{isgn } \xi \ \hat{\sigma}(\xi); \ x]. \tag{4.6}$$

At a general point in the composite solid the components of stress in this case are given by the equations

$$\sigma_{xx}(x, \ y) = \begin{cases} \sigma_{xx}^o(x, \ y) + \sigma_{xx}^1(x, \ y), & (y > 0) \\ \\ \sigma_{xx}^2(x, \ y), & (y < 0) \end{cases} \tag{4.7}$$

$$\sigma_{xy}(x, \ y) = \begin{cases} \sigma_{xy}^o(x, \ y) + \sigma_{xy}^1(x, \ y), & (y > 0) \\ \\ \sigma_{xy}^2(x, \ y), & (y < 0) \end{cases} \tag{4.8}$$

$$\sigma_{yy}(x, \ y) = \begin{cases} \sigma_{yy}^o(x, \ y) + \sigma_{yy}^1(x, \ y), & (y > 0) \\ \\ \sigma_{yy}^2(x, \ y), & (y < 0) \end{cases} \tag{4.9}$$

where the components of stress in the upper half-plane may be found from the equations

$$\sigma_{xx}^1 + \sigma_{yy}^1 = 2\kappa_1^{-1} \ F^*[\phi_1(\xi)e^{-|\xi|y}; \ \xi \to x] \tag{4.10}$$

$$\sigma_{xx}^1 - \sigma_{yy}^1 = \kappa_1^{-1} \ F^*[\{(1 - 2|\xi|y)\phi_1(\xi) + \kappa_1\psi_1(\xi)\}e^{-|\xi|y}; \ \xi \to x] \tag{4.11}$$

$$\sigma_{xy}^1 = -\frac{i}{2}\kappa_1^{-1} \ F^*[\text{isgn } \xi\{(1 - 2|\xi|y)\phi_1(\xi) + \kappa_1\psi_1(\xi)\}e^{-|\xi|y}; \ \xi \to x] \tag{4.12}$$

with

$$\phi_1(\xi) = -2\left(1 + \frac{1}{1 + \kappa_1\Gamma}\right)G_1 i\xi\hat{u}(\xi) + \frac{\kappa_2}{\kappa_2 + \Gamma}\,\hat{\sigma}(\xi) \qquad (4.13)$$

$$\psi_1(\xi) = -2\left(1 + \frac{\kappa_2}{\kappa_2 + \Gamma}\right)G_1 i\xi\hat{u}(\xi) - \frac{\kappa_1}{1 + \kappa_1\Gamma}\,\hat{\sigma}(\xi) \qquad (4.14)$$

while those in the lower half-plane may be found from the equations

$$\sigma_{xx}^2 + \sigma_{yy}^2 = 2\kappa_2^{-1}\,F*[\phi_2(\xi)e^{|\xi|y};\ \xi \to x] \qquad (4.15)$$

$$\sigma_{xx}^2 - \sigma_{yy}^2 = \kappa_2^{-1}\,F*[\{(1 + 2|\xi|y)\phi_2(\xi) + \kappa_2\psi_2(\xi)\}e^{|\xi|y};\ \xi \to x] \qquad (4.16)$$

$$\sigma_{xy}^2 = \tfrac{1}{2}\kappa_2^{-1}\,F*[i\,\mathrm{sgn}\,\xi\{(1 + 2|\xi|y)\phi_2(\xi) + \kappa_2\psi_2(\xi)\}e^{|\xi|y};\ \xi \to x] \qquad (4.17)$$

with

$$\phi_2(\xi) = -\frac{2\kappa_2}{\kappa_2 + \Gamma}\,G_2 i\xi\hat{u}(\xi) - \frac{\kappa_1\Gamma}{1 + \kappa_1\Gamma}\,\hat{\sigma}(\xi) \qquad (4.18)$$

$$\psi_2(\xi) = -\frac{2}{1 + \kappa_1\Gamma}\,G_2 i\xi\hat{u}(\xi) + \frac{\kappa_2\Gamma}{\kappa_2 + \Gamma}\,\hat{\sigma}(\xi). \qquad (4.19)$$

In calculating the auxiliary functions $\phi_1(\xi)$, $\phi_2(\xi)$, $\psi_1(\xi)$, $\psi_2(\xi)$ it is often useful to make use of the formula

$$-i\xi\hat{u}(\xi) = F\left[\frac{\partial u_x^o(x,\ 0)}{\partial x};\ \xi\right]. \qquad (4.20)$$

To illustrate the use of these formulae we shall again consider the stress field due to a point force (X, -Y) acting at the point (0, c) in the upper half-plane.

Again we take $u_x^o(x, y)$ and $u_y^o(x, y)$ to be given by equations (3.24) and (3.25). From equation (3.24) we find that the x-derivative of the tangential component of the surface displacement is given by the formula

$$G_2 \frac{\partial u_x^o(x, 0)}{\partial x} = -\frac{xX\Gamma}{\pi(\kappa_1 + 1)} \left(\frac{\kappa_1}{R^2} - \frac{2c^2}{R^4} \right) - \frac{cY\Gamma}{\pi(\kappa_1 + 1)} \left(\frac{1}{R^2} - \frac{2c^2}{R^4} \right) \tag{4.21}$$

in which we have used the notation

$$R^2 = x^2 + c^2.$$

We then easily derive the formula

$$G_1 i\xi\hat{u}(\xi) = \frac{X}{\sqrt{(2\pi)} \cdot (\kappa_1 + 1)} \{\kappa_1(\text{isgn}\xi) - ci\xi\}e^{-c|\xi|}$$

$$- \frac{Y}{\sqrt{(2\pi)} \cdot (\kappa_1 + 1)} c|\xi| e^{-c|\xi|} \tag{4.23}$$

for $i\xi\hat{u}(\xi)$ and we already have the formula (3.30) for $\hat{\sigma}(\xi)$.

Substituting from equations (4.23) and (3.30) into equations (4.13), (4.14), (4.18) and (4.19) we obtain the formulae

$$\phi_1(\xi) = \frac{X}{\sqrt{(2\pi)} \cdot (\kappa_1 + 1)} \left[\left\{ \frac{(\kappa_1 - 1)\kappa_2}{\kappa_2 + \Gamma} - 2\kappa_1 - \frac{2\kappa_1}{1 + \kappa_1\Gamma} \right\}(\text{isgn}\xi) + 2\left\{ 1 + \frac{1}{1 + \kappa_1\Gamma} - \frac{\kappa_2}{\kappa_2 + \Gamma} \right\} ci\xi \right] e^{-|\xi|}$$

$$- \frac{Y}{\sqrt{(2\pi)} \cdot (\kappa_1 + 1)} \left[\frac{(\kappa_1 + 1)\kappa_2}{\kappa_2 + \Gamma} - 2\left(\frac{\Gamma}{\kappa_2 + \Gamma} + \frac{1}{1 + \kappa_1\Gamma} \right) c|\xi| \right] e^{-c|\xi|}$$

$$\psi_1(\xi) = -\frac{X}{\sqrt{(2\pi)} \cdot (\kappa_1+1)}\left[\left\{\frac{\kappa_1-1}{1+\kappa_1\Gamma} + \frac{2\kappa_2}{\kappa_2+\Gamma} + 2\right\}\kappa_1(isgn\xi) - 2\left\{1 + \frac{\kappa_1}{1+\kappa_1\Gamma} + \frac{\kappa_2}{\kappa_2+\Gamma}\right\}ci\xi\right]e^{-c|\xi|}$$

$$+ \frac{Y}{\sqrt{(2\pi)}(\kappa_1+1)}\left[\frac{\kappa_1(\kappa_1+1)}{1+\kappa_1\Gamma} + 2\left\{1 + \frac{\kappa_1}{1+\kappa_1\Gamma} + \frac{\kappa_2}{\kappa_2+\Gamma}\right\}|c|\xi\right]e^{-c|\xi|}$$

$$\phi_2(\xi) = -\frac{X\Gamma}{\sqrt{(2\pi)}\cdot(\kappa_1+1)}\left[(isgn\xi)\left\{\frac{2\kappa_1\kappa_2}{\kappa_2+\Gamma} + \frac{\kappa_1(\kappa_1-1)}{1+\kappa_1\Gamma}\right\} - 2ci\xi\left\{\frac{\kappa_2}{\kappa_2+\Gamma} + \frac{\kappa_1}{1+\kappa_1\Gamma}\right\}\right]e^{-c|\xi|}$$

$$+ \frac{Y\Gamma}{\sqrt{(2\pi)}\cdot(\kappa_1+1)}\left[\frac{\kappa_1(\kappa_1+1)}{2(1+\kappa_1\Gamma)} + \left\{\frac{\kappa_1}{1+\kappa_1\Gamma} + \frac{\kappa_2}{\kappa_2+\Gamma}\right\}c|\xi|\right]e^{-c|\xi|}$$

$$\psi_2(\xi) = -\frac{X\Gamma}{\sqrt{(2\pi)}\cdot(\kappa_1+1)}\left[\left\{\frac{2\kappa_1}{1+\kappa_1\Gamma} - \frac{(\kappa_1-1)\kappa_2}{\kappa_2+\Gamma}\right\}isgn\xi - \left(\frac{1}{1+\kappa_1\Gamma} - \frac{\kappa_2}{\kappa_2+\Gamma}\right)2ci\xi\right]e^{-c|\xi|}$$

$$- \frac{Y\Gamma}{\sqrt{(2\pi)}(\kappa_1+1)}\left[\frac{(\kappa_1+1)\kappa_2}{\kappa_2+\Gamma} + \left(\frac{\kappa_2}{\kappa_2+\Gamma} - \frac{1}{1+\kappa_1\Gamma}\right)2c|\xi|\right]e^{-c|\xi|} \quad .$$

For example, in the case in which $X = 0$, equation (4.15) gives

$$\frac{1}{2}(\sigma_{xx}^2 + \sigma_{yy}^2)$$

$$= \frac{Y\Gamma}{\sqrt{(2\pi)}(\kappa_1+1)\kappa_2} F^*\left[\left\{\frac{\kappa_1(\kappa_1+1)}{2(1+\kappa_1\Gamma)} + \left(\frac{\kappa_1}{1+\kappa_1\Gamma} + \frac{\kappa_2}{\kappa_2+\Gamma}\right)c|\xi|\right\}e^{-|\xi|(c-y)}; \xi \to x\right]$$

from which it follows that

$$\frac{1}{2}(\sigma_{xx}^2 + \sigma_{yy}^2) = -\frac{Y}{r_1}\left(\gamma_1\sin\theta_1 + \frac{c}{r_1}\gamma_2\cos 2\theta_1\right)$$

where the constants γ_1 and γ_2 are defined by the equations

$$\gamma_1 = \frac{\kappa_1}{2\pi\kappa_2(1 + \kappa_1\Gamma)}, \quad \gamma_2 = \frac{1}{2\pi\kappa_2(\kappa_1 + 1)}\left[\frac{\kappa_1}{1 + \kappa_1\Gamma} + \frac{\kappa_2}{\kappa_2 + \Gamma}\right]$$

and r_1, θ_1 are defined by the first equation of the pair (3.35).

Similarly from equations (4.16) and (4.17) we deduce that

$$\frac{1}{2}(\sigma_{xx}^2 - \sigma_{yy}^2) = -\frac{\gamma}{r_1}\left[\gamma_1\sin3\theta_1 - \gamma_3\sin\theta_1 + \frac{c}{r_1}(\gamma_4\cos2\theta_1 + 2\gamma_2\cos4\theta_1) - \frac{4c^2}{r_1^2}\gamma_2\sin3\theta_1\right]$$

$$\sigma_{xy}^2 = \frac{\gamma}{r_1}\left[\gamma_1\cos3\theta_1 - \gamma_3\cos\theta_1 - \frac{c}{r_1}(\gamma_4\sin2\theta_1 + 2\gamma_2\sin4\theta_1) - \frac{4c^2}{r_1^2}\gamma_2\cos3\theta_1\right]$$

where the constants γ_3 and γ_4 are defined by the equations

$$\gamma_3 = \frac{\kappa_2}{2\pi(\kappa_2 + \Gamma)}, \quad \gamma_4 = \frac{1}{\pi\kappa_2(\kappa_1 + 1)}\left[\frac{\kappa_1^2 + 2\kappa_2}{1 + \kappa_1\Gamma} - \frac{\kappa_2(2\kappa_2 + 1)}{\kappa_2 + \Gamma}\right].$$

5. Griffith Crack at the Interface of Two Half-Planes.

We now consider the problem of determining the distribution of stress in the vicinity of the Griffith crack, described by the relations

$$-1 \leq x \leq 1, \quad y = 0,$$

at the interface of the two half-planes: $y > 0$ which is occupied by elastic material with constants G_1, κ_1 and $y < 0$ which is occupied by elastic material with constants G_2, κ_2.

If we assume that the upper and lower faces of the crack are each subjected to a prescribed pressure $p(x)$, we see that inside the crack area we have the conditions

$$\sigma_{yy}(x, 0+) = \sigma_{yy}(x, 0-) = -p(x), \quad |x| < 1. \tag{5.1}$$

$$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-) = 0 \quad |x| > 1, \tag{5.2}$$

and that on the region of the interface outside the crack we have the conditions

$$u_x(x,\ 0+) = u_x(x,\ 0-) \qquad\qquad |x| > 1, \qquad\qquad (5.3)$$

$$u_y(x,\ 0+) = u_y(x,\ 0-), \qquad\qquad |x| > 1, \qquad\qquad (5.4)$$

$$\sigma_{yy}(x,\ 0+) = \sigma_{yy}(x,\ 0-) \qquad\qquad |x| > 1, \qquad\qquad (5.5)$$

$$\sigma_{xy}(x,\ 0+) = \sigma_{xy}(x,\ 0-) \qquad\qquad |x| > 1, \qquad\qquad (5.6)$$

Adopting the notation

$$u_x(x,\ y) = \begin{cases} iF^*[\xi^{-1}\{A_1 - \kappa_1^{-1}(A_1 - B_1)|\xi|y\}e^{-|\xi|y};\ \xi \to x], & y > 0 \\[2ex] iF^*[\xi^{-1}\{A_2 + \kappa_2^{-1}(A_2 + B_2)|\xi|y\}e^{|\xi|y};\ \xi \to x], & y < 0 \end{cases} \qquad (5.7)$$

$$u_y(x,\ y) = \begin{cases} F^*[|\xi|^{-1}\{B_1 - \kappa_1^{-1}(A_1 - B_1)|\xi|y\}e^{-|\xi|y};\ \xi \to x], & y > 0 \\[2ex] F^*[|\xi|^{-1}\{B_2 - \kappa_2^{-1}(A_2 + B_2)|\xi|y\}e^{|\xi|y};\ \xi \to x], & y < 0 \end{cases} \qquad (5.8)$$

we see from equations (3.13), (3.20) that the conditions (5.1), (5.5) together imply that

$$\kappa_1(\kappa_2 - 1)\Gamma A_2 - \kappa_1(\kappa_2 + 1)\Gamma B_2 = \kappa_2(\kappa_1 - 1)A_1 + \kappa_2(\kappa_1 + 1)B_1.$$

Similarly from equations (3.12) and (3.19) we see that the boundary conditions (5.2) and (5.6) are equivalent to the equation

$$\kappa_1(\kappa_2 + 1)\Gamma A_2 - \kappa_1(\kappa_2 - 1)\Gamma B_2 = -\kappa_2(\kappa_1 + 1)A_1 - \kappa_2(\kappa_1 - 1)B_1$$

Solving these equations for A_2 and B_2 in terms of A_1 and B_1 we find that

$$\kappa_1\Gamma A_2(\xi) = -\frac{1}{2}(\kappa_1\kappa_2 + 1)A_1(\xi) - \frac{1}{2}(\kappa_1\kappa_2 - 1)B_1(\xi) \qquad\qquad (5.9)$$

$$\kappa_1\Gamma B_2(\xi) = -\frac{1}{2}(\kappa_1\kappa_2 - 1)A_1(\xi) - \frac{1}{2}(\kappa_1\kappa_2 + 1)B_1(\xi) \qquad\qquad (5.10)$$

from which we deduce immediately that

$$\kappa_2^{-1}(A_2 + B_2) = -\Gamma^{-1}(A_1 + B_1)$$

Now from equations (5.7) and (5.8) we see that the boundary conditions (5.3) and (5.4) are equivalent to the conditions

$$F*[\xi^{-1}\{A_1(\xi) - A_2(\xi)\}; x] = 0, \qquad |x| > 1,$$

$$F*[|\xi|^{-1}\{B_1(\xi) - B_2(\xi)\}; x] = 0, \qquad |x| > 1,$$

respectively, and using equations (5.9), (5.10) we may reduce these in turn to

$$F*[i\xi^{-1}\{[\kappa_1\Gamma + \tfrac{1}{2}(\kappa_1\kappa_2 + 1)]A_1(\xi) + \tfrac{1}{2}(\kappa_1\kappa_2 - 1)B_1(\xi)\}; x] = 0, \quad |x| > 1 \quad (5.11)$$

$$F*[|\xi|^{-1}\{\tfrac{1}{2}(\kappa_1\kappa_2 - 1)A_1(\xi) + [\kappa_1\Gamma + \tfrac{1}{2}(\kappa_1\kappa_2 + 1)]B_1(\xi)\}; x] = 0, \quad |x| > 1. \quad (5.12)$$

Using equation (2.13) we see that the condition (5.1) is equivalent to

$$F*[(\kappa_1 - 1)A_1(\xi) + (\kappa_1 + 1)B_1(\xi); x] = \kappa_1 G_1^{-1}p(x), \quad |x| < 1 \quad (5.13)$$

and that using equation (5.12) that (5.2) is equivalent to

$$F*[i\operatorname{sgn}\xi\{(\kappa_1 + 1)A_1(\xi) + (\kappa_1 - 1)B_1(\xi); x] = 0, \qquad |x| < 1. \quad (5.14)$$

If we now express $A_1(\xi)$ and $B_1(\xi)$ in terms of two new functions $\phi(\xi)$ and $\psi(\xi)$ through the equations

$$\kappa_1(\kappa_2 + \Gamma)(1 + \kappa_1\Gamma)A_1(\xi) = [\kappa_1\Gamma + \tfrac{1}{2}(\kappa_1\kappa_2 + 1)]\phi(\xi) - \tfrac{1}{2}(\kappa_1\kappa_2 - 1)\psi(\xi) \quad (5.15)$$

$$\kappa_1(\kappa_2 + \Gamma)(1 + \kappa_1\Gamma)B_1(\xi) = -\tfrac{1}{2}(\kappa_1\kappa_2 - 1)\phi(\xi) + [\kappa_1\Gamma + \tfrac{1}{2}(\kappa_1\kappa_2 + 1)]\psi(\xi) \quad (5.16)$$

we may reduce the equations (5.11) through (5.14) to the set of simultaneous dual integral equations

$$F*[\alpha\phi(\xi) + \beta\psi(\xi); x] = f(x), \qquad |x| < 1$$

$$F*[i\text{sgn}\xi\{\beta\phi(\xi) + \alpha\psi(\xi)\}; x] = 0, \qquad |x| < 1 \qquad (5.17)$$

$$F*[i\xi^{-1}\phi(\xi); x] = 0, \qquad |x| > 1$$

$$F*[|\xi|^{-1}\psi(\xi); x] = 0, \qquad |x| > 1$$

in which α and β are constants defined by the equations

$$\alpha = (\kappa_1 - 1)\Gamma - (\kappa_2 - 1), \quad \beta = (\kappa_1 + 1)\Gamma + (\kappa_2 + 1), \qquad (5.18)$$

and the function $f(x)$ may be calculated from the prescribed function $p(x)$ by means of the equation

$$f(x) = \{G_1(\kappa_2 + \Gamma)(1 + \kappa_1\Gamma)\}^{-1}p(x). \qquad (5.19)$$

It should be noted that

$$\beta^2 - \alpha^2 = 4(\kappa_2 + \Gamma)(1 + \kappa_1\Gamma). \qquad (5.20)$$

These equations may be solved by standard methods; their solution and its implications for fracture mechanics will be considered in a future publication.

It is of interest to note that the solution of these equations may be reduced to that of a singular integral equation by the following device:

If we integrate, with respect to x, the first two equations of the set (5.17) we see that they can be written in the alternative form

$$\alpha F*[i\xi^{-1}\phi(\xi); x] + \beta F*[i\text{sgn}\xi.|\xi|^{-1}\psi(\xi); x] = F(x) + C_1,$$

$$\alpha F*[i\text{sgn}\xi.i\xi^{-1}\phi(\xi); x] - \beta F*[|\zeta|^{-1}\psi(\xi); x] = C_2,$$

where C_1 and C_2 are arbitrary constants and $F'(x) = f(x)$. Now if we let

$$F*[i\xi^{-1}\phi(\xi); x] = \phi(x)H(1 - x),$$

$$F*[|\xi|^{-1}\psi(\xi); x] = \Psi(x)H(1 - x),$$

we automatically satisfy the third and fourth equations of the set (5.17),
and can write these last two equations in the form

$$\alpha\phi(x) + \frac{\beta}{\pi} \int_{-1}^{1} \frac{\Psi(t)dt}{x - t} = F(x) + C_1, \qquad |x| < 1$$

$$\frac{\alpha}{\pi} \int_{-1}^{1} \frac{\Phi(t)dt}{x - t} - \beta\Psi(x) = C_2, \qquad |x| < 1$$

in integrals being interpreted as Cauchy principal values. If we write

$$\alpha\phi(x) - i\beta\Psi(x) = X(x), \quad C_1 + iC_2 = \gamma$$

we may write this pair of singular integral equations as the single equation

$$X(x) + \frac{i}{\pi} \int_{-1}^{1} \frac{X(t)dt}{x - t} = F(x) + \gamma.$$

6. Elastic Half-Space Bonded to a Rigid Foundation.

We now consider the special case when the lower half-plane is rigid.

In the elastic medium $y > 0$ with constants G and κ, we may take as the displacement field

$$u_x(x, y) = u_x^0(x, y) + F*[i\xi^{-1}\{A - \kappa^{-1}(A - B)|\xi|y\}e^{-|\xi|y}; \xi \to x] \qquad (6.1)$$

$$u_y(x, y) = u_y^0(x, y) + F*[|\xi|^{-1}\{B - \kappa^{-1}(A - B)|\xi|y\}e^{-|\xi|y}; \xi \to x] \qquad (6.2)$$

for which

$$\sigma_{xy}(x, 0+) = \sigma_{xy}^0(x, 0+) - \kappa^{-1}GF*[i\,\text{sgn}\,\xi\{(\kappa + 1)A(\xi) + (\kappa - 1)B(\xi)\}; x] \qquad (6.3)$$

$$\sigma_{yy}(x, 0+) = \sigma_{yy}^0(x, 0+) - \kappa^{-1}GF*[(\kappa - 1)A(\xi) + (\kappa + 1)B(\xi); x]. \qquad (6.4)$$

Here again, the displacement field (u_x^o, u_y^o) has the correct singularities to describe the distribution of body forces in the elastic half-space.

If the elastic medium is <u>bonded</u> to the rigid foundation we have the conditions

$$u_x(x, 0+) = u_y(x, 0+) = 0$$

which from equations (6.1) and (6.2) are equivalent to the pair of formulae

$$A(\xi) = i\xi\hat{u}(\xi)$$

$$B(\xi) = -|\xi|\hat{v}(\xi)$$

with

$$\hat{u}(\xi) = F[u_x^o(x, 0+); \xi], \quad \hat{v}(\xi) = F[u_y^o(x, 0+); \xi]$$

for the determination of the functions $A(\xi)$ and $B(\xi)$. Equations (6.1) and (6.2) become

$$u_x(x, y) = u_x^o(x, y) - F*[\{1 - \kappa^{-1}|\xi|y\}e^{-|\xi|y}\hat{u}(\xi); \xi \to x]$$

$$- \kappa^{-1}y \, F*[i\xi e^{-|\xi|y}\hat{v}(\xi); \xi \to x] \qquad (6.5)$$

$$u_y(x, y) = u_y^o(x, y) - \kappa^{-1}y \, F*[i\xi e^{-|\xi|y}\hat{u}(\xi); \xi \to x]$$

$$- F*[\{1 + \kappa^{-1}|\xi|y\}e^{-|\xi|y}\hat{v}(\xi); \xi \to x] \qquad (6.6)$$

Using the convolution theorem for Fourier transforms we may write these equations in the form

$$u_x(x, y) = u_x^o(x, y) - \int_{-\infty}^{\infty} \{G_1(x - t, y) - G_2(x - t, y)\}u_x^o(t, 0)dt$$

$$- \int_{-\infty}^{\infty} G_3(x - t, y)u_y^o(t, 0)dt, \qquad (6.7)$$

$$u_y(x, y) = u_y^o(x, y) - \int_{-\infty}^{\infty} G_3(x - t; y)u_x^o(t, 0)dt$$

$$- \int_{-\infty}^{\infty} \{G_1(x - t, y) + G_2(x - t, y)\}u_y^o(t, 0)dt \qquad (6.8)$$

in which the functions G_1, G_2, and G_3 are defined by the equations

$$G_1(x, y) = \frac{y}{\pi(x^2 + y^2)}, \quad G_2(x, y) = \frac{y(y^2 - x^2)}{\kappa\pi(x^2 + y^2)^2}, \quad G_3(x, y) = \frac{2xy^2}{\kappa\pi(x^2 + y^2)^2}. \qquad (6.9)$$

If there is a Griffith crack $|x| < 1$ at the interface between the elastic half-space and the rigid foundation the basic boundary value problem to be solved is:

$$\sigma_{yy}(x, 0) = -p(x), \quad |x| < 1$$

$$\sigma_{xy}(x, 0) = -q(x), \quad |x| < 1$$

$$u_x(x, 0) = u_y(x, 0) = 0, \quad |x| > 1$$

In this case the solution is given by equations (6.1), (6.2) in which $A(\xi)$ and $B(\xi)$ satisfy the dual integral equations

$$F^*[(\kappa - 1)A(\xi) + (\kappa + 1)B(\xi); x] = f(x), \qquad |x| < 1$$

$$F^*[i\,\mathrm{sgn}\xi\{(\kappa + 1)A(\xi) + (\kappa - 1)B(\xi); x] = g(x), \qquad |x| < 1$$

$$F^*[i\xi^{-1}A(\xi); x] = 0, \qquad |x| > 1$$

$$F^*[|\xi|^{-1}B(\xi); x] = 0, \qquad |x| > 1$$

where

$$f(x) = \kappa G^{-1}\{p(x) + \sigma_{yy}^o(x, 0+)\}$$

$$g(x) = \kappa G\{q(x) + \sigma_{xy}^o(x, 0+)\}$$

This is precisely the problem encountered in §5.

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